On The Solution of Pension Fund Investment and Utility Optimization with Backward Stochastic PDE

Onwukwe Ijioma
Mathematics Department, Abia State University, Nigeria
onwukweijioma@abiastateuniversity.edu.ng

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ABSTRACT:

The problem of maximizing expected utility of pension fund investment returns from terminal wealth is considered. This observable stochastic control problem turns out to be a risky sensitive type especially with random economic factors. Under some assumptions , we established the BSPDE with a corresponding strategy that is optimal.

Keywords: *Pension funds, Utility Optimization, Doléans measure, Backward PDE*

1. Introduction:

The fundamental idea of the pension fund manager is to maximize the investor's funds using different investment strategies. There are remarkable works on the management and optimization of pension funds. The process of managing pension funds involves a continuously observe risk exposure and rebalancing the portfolios. There are three major combined processes namely: accumulation of contributions , the investment returns of contributions and the presence of minimum guarantee on the benefits at retirement or after retirement [1].
Generally, portfolio optimization, hedging and effective pricing of derivative are primarily problems associated with mathematical finance.

The function of the pension fund manager (PFM) is to optimize
$$\mathrm{E}[\mathcal{U}(\mathrm{X}_T^{x,\pi})] \quad over\ all\ \ \pi\ from\ the\ given\ \Lambda\ of\ \ \ staregies \tag{1}$$
$where\ \ \mathrm{X}_l^{x,\pi} = x + \int_0^t \pi_u dS_u$ is the wealth process starting from the initial capital x , obtained by the self financing trading process $\pi\ and\ \Lambda$ is class of admissible strategies.

Given that $\mathcal{U}$ is a function which measures a hedging error, where we assume that $\mathcal{U}(x)$ is strictly convex for each τ and we have a minimization problem of the form
$$Min\ \mathrm{E}[\mathcal{U}(\mathrm{X}_T^{x,\pi})] \text{ on } \pi \in \Lambda \tag{2}$$

From [2], a backward stochastic PDE for value function
$$V(t,x) = \underset{\pi\in\Lambda}{essinf}\,\mathrm{E}(\mathcal{U}(x + \int_t^T dS_u)/F_t) \tag{3}$$
of (2) was derived and in terms of solutions of this equation with optimal strategies.

The issue of utility maximization problem was first studied by Merton (1971) in the classical Black-Scholes model. Pliska (1986), Cox and Huang (1989) and Karatzas et al (1987) showed that the optimal portfolio of the utility maximization problem is the same as the density of the martingale measure for complete measure, whereas, He and Pearson(1991) and Karatzas et all (1987) showed for the case of incomplete markets, where he employed Ito process to describe the incomplete market , which gives a duality characterization of optimal portfolios obtained by the set martingale measure and the solution of the primal by convex duality.

Our goal is to find a semi martingale Bellman equation which is a stochastic version of the Bellman equation. We consider a continuous semi martingale defined on a probability space satisfying the usual condition.

Let $\Omega(s)$ *be an* R^{α} - valued continuous semi martingale. The process Ω is the discounted price evolution of the α-risky assets in a financial market with riskless assets. Since Ω is continuous, the existence of an equivalent martingale measure implies a structured condition is fulfilled.

It implies that Ω assumed a decomposition structure.

$$\Omega_t = M_t + \int_0^t d(M)_s\,\lambda_s \text{ , and } \int_0^t \lambda_s^t d(M)_s \lambda_s < \infty \ \forall\ t \tag{4}$$

Where M is a CLM and λ is a predictable R^{α}- valued process.

The utility function $\mathcal{U}$ under consideration is a mapping $R = (0,\infty) \to R$ under the assumption that its continuously differentiable strictly concave and satisfy the Inada conditions.

$$U'(0) = \lim_{x\to 0} U'(x) = \infty \tag{5}$$

$$U'(\infty) = \lim_{x\to 0} U'(x) = 0 \tag{6}$$

Let π_x be the class of predictable Ω_t process π such that , the wealth process yields

$$X_t^{\pi,x} = x + \int_0^t \pi_u d\Lambda_u \geq 0 \quad \forall\ t \in [0,T] \tag{7}$$

From (2) , the value function is defined as

$$V(t,x) = \underset{\pi\in\Pi_x}{esssup}\ \mathrm{E}\left(\mathcal{U}(\mathrm{x} + \int_{\mathrm{t}}^{\mathrm{T}} \pi_{\mathrm{u}} \mathrm{d}\Lambda_{\mathrm{u}}\right)/\mathcal{F}_{\mathrm{t}}) \tag{8}$$

The Itô-Ventzel formula is used in the derivation of the Bellman equation for the value function of the diffusion processes. In the stated theorem 1, the value function satisfies the backward stochastic partial differential equation (BSPDE).

$$V(t,x) = V(0,x) + \frac{1}{2}\int_0^t \frac{(\varphi_x(s,x) + \lambda(s)V_x(s,x))^2}{V_{xx}(s,x)} d(M)_s + \int_0^t \varphi(s,x) dM_s + L(t,x) \tag{9}$$

with the boundary condition $V(T,x) = U(x), where\ \int_0^t \varphi(s,x)dM_s + L(t,x)$ is the martingale component of $V(t,x)$. In particular π^* is optimal if the corresponding wealth process $X^{\pi*}$ is a solution of the forward SDE

$$X_t^{\pi*} = X_0^{\pi*} - \int_0^t \frac{\varphi_x(u,X_u^{\pi*}) + \lambda(u)V_x(u,X_u^{\pi*})}{V_{xx}(s,X_u^{\pi*})} dS_u \tag{10}$$

we need to show that for standard utility functions and the conditions to be stated in the theorem 1 are satisfied and therefore guarantee the existence of a solution to the backward equations.

2. Assumptions:

We consider an incomplete financial market where the dynamics of asset prices are governed by an R^{α} - valued continuous semi martingale S defined on the usual filtered probability space $(\Omega, \mathcal{F}, \mathcal{F} = (\mathcal{F}_t, t \in [0,T]), \mathcal{P})$ satisfying the usual conditions.

By the definition of Bounded Mean Oscillation (BMO)- martingale , the square integrable continuous martingale $\mathcal{M}$ belongs to the class BMO, if given $c > 0$ such that $E(\langle M\rangle_T - \langle M\rangle_\tau | \mathcal{F}_\tau) < c\ \ a.s. \forall\tau \geq 0.$ A strictly positive uniformly integrable martingale $\mathbb{Z}$ satisfies the Muckenhoupt inequality defined by $\mathcal{A}_\alpha(\mathcal{P}), 1 < \alpha < \infty\ \ iff\ E\left(\left(\frac{Z_\tau}{Z_T}\right)^{\frac{1}{\alpha-1}} |\mathcal{F}_\tau\right) < c\ a.s. \forall\tau$

Let π_x be the space of all predictable S-integrable processes π such that the wealth process is nonnegative at any point such that $x + \int_0^t \pi_u dS_u \geq 0 \ \forall\ t \in [0,T]$.

Here, we state the definition of Doléans measures

Definition 1: *Let* $\{S_t: 0 \leq t \leq 1\}$ be a Cadlag submartingale with $S_0 \equiv 0$. Say that a finite (countably-additive) measure μ_s, defined on the predictable sigma-field $\mathcal{P}\ on\ \mathfrak{S}$, the Doléans measure for $S\ if\ \mu(0,\tau] = \mathbb{P}S_\tau\ for\ every\ \tau \in \tau_1$, the set of all $[0,1]$ −valued stopping times for the filtration.

By the property of $[\mathcal{D}]$, $[\mathcal{D}]$ the set of random variables $\{S_t: 0 \leq t \leq 1\}$ is uniformly integrable.

Theorem 1: Every Cadlag submartingale $\{S_t: 0 \leq t \leq 1\}$ with property $[\mathcal{D}]$ has a countably additive Doléans measure μ_s on the predictable sigma-field $\mathcal{P}\ on\ \mathfrak{S} = \Omega \times (0,1]$.

By Doléans measure for an increasing process k. *Let* $\mathcal{U}(x) = \mathcal{U}(\omega, x)$ satisfies the following conditions:

C1: $V(0,x) < \infty$, $for\ \ x > 0$
C2: $\mathcal{U}(\omega, x)$ is twice continuously differentiable and strictly concave for ω.
C3: $For\ \ any\ t\ and\ x\ \exists\ a\ \ strategy\ \pi^*(t,x)\ such\ that$

$$V(t,x) = \ E\left(U(x + \int_t^T (t,x)dS_s\ /\mathcal{F}_t\right). \tag{11}$$

Given that $\mathcal{AE}(U) = \lim_{x\to\infty} sup\frac{xU_x(x)}{U(x)} < 1$ implies that utility function $U(x) < 1$ (12)

For optimality strategies, there exist two distinct strategies $\pi^{*1}\ and\ \pi^{*2}$ such that

$$\frac{1}{2}E\left[\left(x + \int_t^T \pi_s^{*1}\, dS_s\right) / \mathcal{F}_t\right] + \frac{1}{2}E\left[\left(x + \int_t^T \pi_s^{*2}\, dS_s\right) / \mathcal{F}_t\right]$$

$$= E\left[U\left(x + \int_t^T (\tfrac{1}{2}\pi^{*1} + \tfrac{1}{2}\pi^{*2}\, dS_s\right) / \mathcal{F}_t\right] \tag{13}$$

And $\frac{1}{2}U\left(x + \int_t^T \pi_s^{*1}\, dS_s\right) + \frac{1}{2}U\left(x + \int_t^T \pi_s^{*2}\, dS_s\right) = U\left(x + \int_t^T \pi_s^{\sigma}\, dS_s\right)$,$\mathcal{P} - a.s$

$$where\ \pi_s^{\sigma} = \frac{1}{2}\pi^{*1} + \frac{1}{2}\pi^{*2} \tag{14}$$

Strong concavity of U implies $\int_t^T \pi_s^{*1} dS_s = \int_t^T \pi_s^{*2} dS_s$ (15)

3. Formulation Of BSPDE For The Value Function

We state and derive backward stochastic partial differential equations (BSPDE) for the value function associated to the utility maximization problem of pension funds investment.
Define the class of function $J\colon \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}\ \ denote\ by\ Z^{1,2}$. Let the BSPDE defined by

$$J(t,x) = J(0,x) + \frac{1}{2}\int_0^t \frac{(\psi_x(s,x) + \lambda(s)J_x(s,x))'}{J_{xx}(s,x)} d\langle M\rangle_s(\psi_x(s,x) + \lambda(s) + J_x(s,x))$$

$$+ \int_0^t \psi(s,x)dM_s + L(t,x)\ . \tag{16}$$

Satisfying the boundary condition $J(T,x) = U(x)$. (17)
J is a solution of (16) and (17) if
$J(\omega,t,x)$ is twice differentiable for each (ω,t) with the boundary condition (17)

$$\mathcal{P} - a.s.\ \forall\, x \in \mathbb{R},\ B(t,x) = \frac{1}{2}\int_0^t \left(\psi_x(s,x) + \lambda(s)J_x(s,x)\right)' d\langle M\rangle_s(\psi_x + \lambda(s)J_x\,(s,x) \tag{18}$$

Optimality Principle:
Proposition 1. $Let\ V(0,x) < \infty\ \ for\ \ \ some\ \ x$, then

(a) For all $x \in \mathbb{R}, \pi \in \Pi\ \ and\ \ s \in [0,T]$ the process $\left(V\left(t, x + \int_s^t \pi_u dS_u\ ,\ t \ge s\right)\right)$ is a supermartingale admitting an RCLL condition
(b) $\pi^*(s,x)$ is optimal if and only if $\left(V\left(t, x + \int_s^t \pi_u^* dS_u\right),\ t \ge s\right)$ is a martingale
(c) For all $s < t$, $V(s,x) = \underbrace{ess\ sup}_{\pi\in\Pi(s,T)}\ E\ \left(J\left(t, x + \int_s^t \pi_u dS_u\right)/\mathscr{F}_s\right)$ (19)

From the proposition 1, the value process $V(t,x)$ is a supermartingale for any $x \in \mathbb{R}$ which admits the decomposition: $V(t,x) = J(0,x) + A(t,x) + \int_0^t \varphi(s,x)dM_s + M(x)(t,x)$ (20)
where $A_{(x)} \in A_{loc}$,$m(x)$ is a local martingale to M for all $x \in \mathbb{R}_+$ and φ_x is the partial derivative of φ.
Again , $A(t,x) = \int_0^t a(s,x)dK_s$ is a measurable function $a(t,x)$.
Theorem 2: $Let\ V = \mathcal{V}^{1,2}$ with the conditions as in C1- C3. Then, the value function is a solution of BSPDE (16)- (17) :

$$J(t,x) = J(0,x) + \frac{1}{2}\int_0^t \frac{(\psi_x(s,x) + \lambda(s)J_x(s,x))'}{J_{xx}(s,x)} d\langle M\rangle_s(\psi_x(s,x) + \lambda(s) + J_x(s,x))$$

$+ \int_0^t \psi(s,x)dM_s + L(t,x)$, satisfying the boundary condition $J(T,x) = U(x)$.

The strategy π^* is optimal if and only if the corresponding wealth process $X^{\pi*}$ is a solution of the forward SDE (10), such that the process $V(t, X^{\pi*})$ is from the family of random variables $Z_\tau I_{(\tau\le T)}$ extracted from $[\mathcal{D}]$ for all stopping times τ is uniformly integrable.

Proof: $Let\ \ \pi^*(s,x)$ be the optimal strategy. The principle of optimality implies that $\left(V(t, X_t^*(s,x)\right)$,$t \ge s$ is a martingale. Using Ito-Ventzell's formula, we have

$$\int_s^t \left[a(u, X_u^*(s,x)) - g(u, X_u^*(s,x)) + \left| \pi_u^*(s,x) + \frac{V_x(u,X_u^*(s,x))\lambda(u) + \varphi_x(u,X_u^*(s,x))}{V_{xx}(u,X_u^*(s,x))} \right|_{V_u}^2 \right] dK_u = 0 \qquad (21)$$

$\forall\, t \geq s$, $\mathcal{P} - a.s$ where we define $g(s,x) = \frac{1}{2}\frac{|\varphi_x(s,x)+\lambda(s)V_x(s,x)|_{V_s}^2}{V_{xx}(s,x)}$ (see [4]) (22)

It follows that $(a(s,x) - g(s,x))(K_s - K_{s^-}) = 0 \quad \forall s \in [0,T]$.

Therefore, $a(s,x) = g(s,x) \quad a.s$

4. Utility Maximization of Pension Fund Investment

In this section, we drive the value function for the power utility function, with the constraints of optimal strategies for the utility maximization problem.

The goal of the Pension fund manager (PFM) is to optimize the long-term growth of the expected utility of returns. The optimal allocation problem faced by the PFM is different in its objectives from the problem faced by an individual investor having direct access into the investment market.

$Let\ U(x) = x^{\alpha}, \alpha \in (0,1]$, from (1), we recast the power utility problem to a maximization problem as $Max\ E\left(x + \int_0^T \pi_u dS_u\right)^{\alpha}$ $for\ some\ \ \pi \in \Lambda_x, where\ \Lambda_x$ is a class of admissible strategies.

Given the transitory stage within the time interval $[0,T]$, where $t = 0$, the first time of operation of the fund , and the running time period of a given time $s \in [0,T]$, the maximization over the set of admissible strategies defined by

$$\Theta_{ad}(s,x) = \{\theta: [s,T] \times \Omega \to [0,1]\ wrt\ \{\mathcal{F}_t\}_{t\in[s,T]} | X(t;s,x,\theta(\cdot) \geq \ell(t)\ , t \in [s,T])\} \qquad (23)$$

Where $\ell(t)$ is derived from the state variable above the solvency level.

In this case the value function $V(t,x)$ is of the form $x^{\alpha}V_t$, where V_t is a special semimartingale. Since Λ_x is a cone for some $x > 0$ where the strategy $\pi \in \Lambda_x\ iff\ \frac{\pi}{x} \in \Lambda_1$ then $V(t,x) = \underset{\pi\in\Lambda_x}{esssup}\ E\left(\left(x + \int_t^T \pi_u dS_u\right)^{\alpha} / \mathcal{F}_t\right)$

$$= x^{\alpha} \underset{\pi\in\Lambda_x}{esssup}\ E\left(\left(1 + \int_t^T \frac{\pi_u}{x} dS_u\right)^{\alpha} / \mathcal{F}_t\right) = x^{\alpha}V_t \qquad (24)$$

Where $V_t = \underset{\pi\in\Lambda_1}{ess\ sup}\ E\left(\left(1 + \int_t^T \pi_u dS_u\right)^{\alpha} / \mathcal{F}_t\right)$ (25)

is the supermartingale by optimality principle.

5. Conclusion

If $U(x) = x^{\alpha}, \alpha \in (0,1)$ then, the value function $V(t,x)$ is of the form $x^{\alpha}V_t$, V_t satisfies the backward stochastic differential equation (BSDE)

$$V_t = V_0 + \frac{q}{2}\int_0^t \frac{(\varphi_s + \lambda_s V_s)'}{V_s} d\langle M\rangle_s (\varphi_s + \lambda_s V_s) + \int_0^t \varphi_s\, dM_s + L_t\, , V_T = 1 \qquad (26)$$

Where $q = \frac{\alpha}{\alpha - 1}$ and L is a local martingale strongly orthogonal to M. The optimal wealth process is a solution of the linear equation $X_t^* = X - (q-1)\int_0^t \frac{\varphi_u + \lambda_u V_u}{V_u} X_u^* dS_u$ (27)

And the optimal strategy is finally resolved as

$$\pi_t^* = -\alpha(q-1)\varepsilon_t\left(-(q-1)\left(\frac{\varphi}{V} + \lambda\right)\cdot S\right)\left(\frac{\varphi_t}{V_t} + \lambda_t\right)$$

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